Multimodal nature of edge waves in thick plates and shells and their potential applications in non-destructive testing: theory and experimental verification

Maria V Wilde[1], Artem A Eremin[2] and Mikhail V Golub[2]

1 Faculty of Mathematics and Mechanics, Saratov State University, Saratov, Russian Federation; e-mail: mv_wilde@mail.ru

2 Institute for Mathematics, Mechanics and Informatics, Kuban State University, Krasnodar, Russian Federation, e-mails: m_golub@inbox.ru, eremin_a_87@mail.ru

Abstract

Results of theoretical investigations of edge waves (EW) in plates and shells on the basis of the 3D elastodynamic theory are summarized in order to show the applicability of these waves for the purposes of NDT. It is shown that there is a system of EW including fundamental modes and higher order ones, so the multimodal nature of EW is revealed. The solution of boundary value problems describing the propagation of EW in a semi-infinite plate or in a semi-infinite cylindrical shell is obtained using the semi-analytical method based on the normal modes expansion. In this method, all the relations are satisfied analytically except the boundary conditions on the edge for which the effective numerical procedure is proposed. For the case of a thick plate with free faces, the results of experimental verification are presented. The wave motion in a thick aluminium plate was excited by a piezoelectric transducer and registered by scanning Laser Doppler vibrometry. Comparison of computed spectral properties with ones of the measured signals shows a good agreement between theory and experiment.

1. Introduction

Surface elastic waves, first predicted theoretically by Rayleigh in 1885, are widely used in non-destructive testing (NDT). Analogous phenomena described by 2D theories of thin plates are edge waves (EW) guided by the edge of a plate. They are investigated since the 1950s, but only in the recent studies [1-6] relying on 3D elastodynamic theory it has been revealed infinite number of higher order EW. Therefore, one can say that EW are multimodal in a way similar to Lamb waves. The high-order EW have a wide range of potential applications for the purposes of NDT since they allow detecting defects much smaller than the thickness of a plate and exist practically in all possible cases of boundary conditions on the faces.

The goal of the present paper is to summarize the results of theoretical investigations of EW in plates and shells and to present them in the form allowing estimation of their potential as an NDT-tool. In doing so, the contribution focuses on computational results rather than on the complicated mathematical aspects of the problem and presents all the properties of the EW system, including symmetric and antisymmetric ones, in the frequency range from zero to infinity. Moreover, different types of boundary conditions on the faces and edge providing the existence of EW are indicated. The most interesting results for plates and shells with a not completely free surface are presented.

Experimental verification of the theoretical results for the case of a plate with free surfaces was carried out on the aluminium plate-like specimen. Comparison of theoretical predictions with the experimental data is presented in Section 5 of this paper.

2. Statement of the problem

Consider stationary vibrations of a semi-infinite isotropic elastic plate referred to Cartesian coordinates as shown in Figure 1,*a*. The displacement vector $\boldsymbol{u}$ can be expressed as decomposition

$$\boldsymbol{u} = \text{grad}\,\varphi + \text{rot}\,\boldsymbol{\psi}, \tag{1}$$

where φ and $\boldsymbol{\psi}$ are wave potentials. If the time dependence is assumed in the form $e^{-\mathrm{i}\omega t}$, equations for the potentials are

$$\Delta\varphi + \frac{\omega^2}{c_1^2}\varphi = 0, \;\; \Delta\boldsymbol{\psi} + \frac{\omega^2}{c_2^2}\boldsymbol{\psi} = 0, \tag{2}$$

where c_1 and c_2 are the longitudinal and transverse wave speeds, respectively. The stresses σ_{ij} can also be expressed in terms of φ and $\boldsymbol{\psi}$ (see, e.g., [4]). In order to study EW, the wave potentials are assumed in the following form

$$\varphi = \Phi(x_2, x_3)e^{\mathrm{i}\xi x_1}, \quad \boldsymbol{\psi} = \boldsymbol{\Psi}(x_2, x_3)e^{\mathrm{i}\xi x_1} \tag{3}$$

and non-trivial solutions satisfying some homogeneous boundary conditions (BC) and decay condition as $x_3 \to -\infty$ are sought for. Two types of BC on the faces are considered: free faces

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = 0 \;\text{ at }\; x_2 = \pm H/2, \tag{4}$$

and fixed ones

$$u_1 = u_2 = u_3 = 0 \;\text{ at }\; x_2 = \pm H/2. \tag{5}$$

On the edge one of the following types of BC is imposed:

$$\sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \;\text{ at}\; x_3 = 0, \tag{6}$$

$$\sigma_{31} = u_2 = \sigma_{33} = 0 \;\text{ at}\; x_3 = 0, \tag{7}$$

$$u_1 = \sigma_{32} = \sigma_{33} = 0 \;\text{ at}\; x_3 = 0. \tag{8}$$

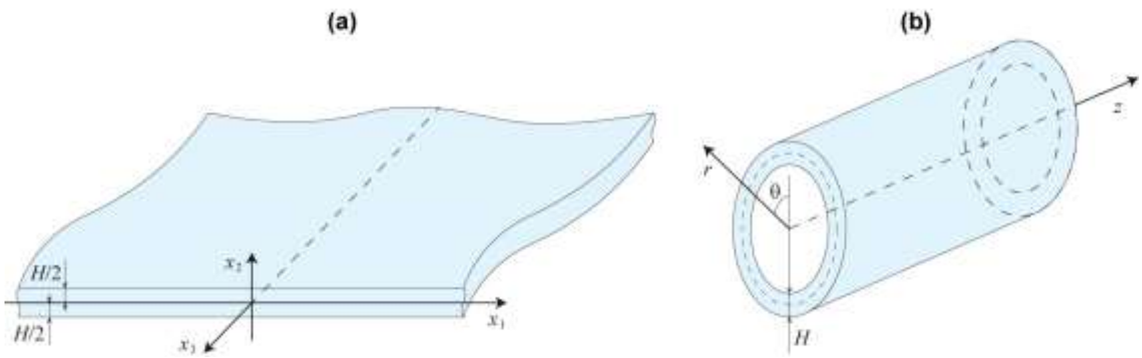

Figure 1. Geometry of the problem: (a) semi-infinite plate; (b) semi-infinite cylindrical shell

The complex amplitudes Φ and $\mathbf{\Psi}$ are presented as a linear combination of solutions satisfying BC (4) or (5) (normal modes). The coefficients of the normal modes expansion are defined by satisfying BC on the edge employing collocation method (see [4] for more details). It is more convenient not to seek the zeros of the corresponding determinant but consider the problem of forced vibrations with non-homogeneous BC on the edge. The sought-for eigenvalues correspond to the resonance frequencies which are easier to found numerically. In the case of complex eigenvalues, real and imaginary parts can be defined as the resonance frequency and the width of the resonant peak, respectively. In doing so, one can avoid solving dispersion equations for normal modes with complex frequency parameter. The recent investigations of complex poles corresponding to EW in non-stationary problem [7] show that the approach described above is suitable for studying of EW dispersion properties.

To investigate EW in shells a semi-infinite cylindrical shell is considered using cylindrical coordinates as shown in Figure 1,b. The decomposition (1) and equations (2) are valid in this case with differential operators written in cylindrical coordinates. BC on the faces of the shell are

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \ \text{at}\, r = R \pm H/2 \quad (9)$$

(free faces). On the edge two types of boundary conditions are considered: free edge

$$\sigma_{zr} = \sigma_{z\theta} = \sigma_{zz} = 0 \ \text{at}\, z = 0, \quad (10)$$

and edge fixed in radial direction

$$u_r = \sigma_{z\theta} = \sigma_{zz} = 0 \ \text{at}\, z = 0. \quad (11)$$

The solution describing EW has the following form

$$\varphi = \Phi(r,z)e^{\mathrm{i}p\theta}, \quad \boldsymbol{\psi} = \mathbf{\Psi}(r,z)e^{\mathrm{i}p\theta}, \quad (12)$$

where the wavenumber is defined as $\xi = p/R$. The method of the solution remains the same as for the plate, but in this case the normal modes are of a hollow cylinder. All the results presented below in Sections 3 and 4 are computed for a material with the value of Poisson ratio $\nu = 0.25$.

3. Edge waves in plates

In this section results of EW investigation in plates are considered. First, the solutions for extensional and bending EW described by 2D theory of plate extension and Kirchhoff theory of plate bending are studied, respectively. When considering the plate as a three-dimensional body, the first wave is referred to as a symmetric one and the second as an antisymmetric. The calculations based on the 3D elastodynamic theory confirm the existence of these waves but show that the plate theories allow obtaining their dispersion characteristics with acceptable accuracy only in a certain low-frequency range. When frequency grows, the behaviour of dispersion curves changes in a qualitative way. At high frequencies both waves become a pair of wedge waves, i.e. waves localized in the vicinity of the wedges $x_2 = \pm H/2, x_3 = 0$. For the antisymmetric wave these results agree with the results obtained via another computation method described in [8].

Investigations of edge resonances in [4] showed that there exist infinite spectra of them both in symmetric and antisymmetric cases. This fact suggests the existence of infinite spectra of EW since the problem for edge resonance coincides with the problem under consideration at $\xi = 0$. By analogy with Lamb waves, the notations ES_k and EA_k with k=1,2,... for the EW whose dispersion curves starting from kth edge resonance frequency are introduced. The waves corresponding to EW in 2D plate theories are denoted as ES_0 and EA_0. Since their frequencies tend to zero as the wavenumber tends to zero, they are called fundamental waves and, correspondingly, ES_k and EA_k are higher order waves. For the latter, the simple heuristic formulae describing the behaviour of dispersion curves as $\xi \to \infty$ are obtained in [4,5]. For the case of non-zero Poisson ratio they are

$$\omega_n^{\mathrm{s}} \approx c_R\sqrt{\xi^2 + (n+0.5)^2}, \quad \omega_n^{\mathrm{a}} \approx c_R\sqrt{\xi^2 + (n+1)^2}, \quad n = 1,2,..., \qquad (13)$$

where c_R is Rayleigh wave speed, the superscripts "s" and "a" denote symmetric and antisymmetric waves. Relations (13) also show the existence of higher order EW spectrum with c_R as a limit velocity as $\xi \to \infty$. In numerical investigations, the dispersion curves were traced starting from the edge resonance frequencies at $\xi = 0$. The results are presented in Figure 2,*a*. At sufficiently large ξ a good agreement with approximations (13) is achieved (see [4,5] for more details). In addition, one more EW appears both in symmetric and antisymmetric cases at some value of ξ (see Figure 2,*a*). For these waves notations $ES_{0.5}$ and $EA_{0.5}$ were introduced.

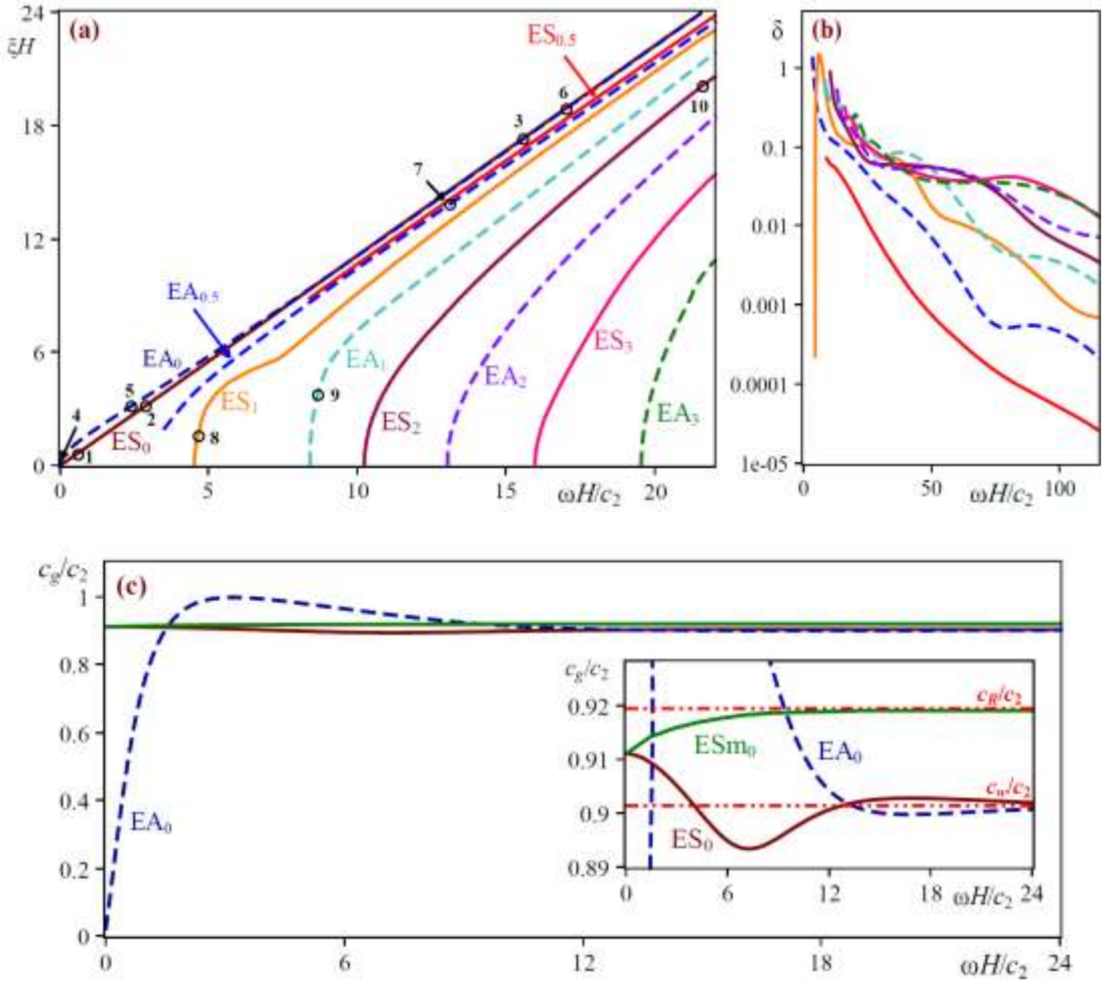

Figure 2. Edge waves in a plate with free faces: (a) dispersion curves; (b) logarithmical decrement; (c) group velocities for fundamental waves

With some exceptions (see [2]) the frequencies of all higher order EW are complex. The imaginary parts describe attenuation of EW because of radiation with Lamb waves propagating from the edge. The link between EW and Lamb waves arises due to partial transformation of Rayleigh wave in bulk waves when reflecting from the faces of the plate. In Figure 2,b the logarithmical decrements (LD) $\delta = 2\pi\,\Gamma/\omega$ are plotted (here Γ is the width of the resonant peak coinciding approximately with the imaginary part of the eigenfrequency). At low and moderate values of thickness-frequency product, the attenuation of higher order waves is strong, so fundamental waves could be the most important tool in applications. The group velocities of fundamental waves are shown in Figure 2,c, where c_w is the wedge wave speed.

In Figure 3 the density of the energy flux through the cross-section $x_1 = const$ averaged over the period of oscillations

$$P_1 = -\frac{\omega}{2}\mathrm{Im}(u_j\sigma_{j1}^*) \qquad (14)$$

is plotted for the points indicated in Figure 1,a. These graphs show the distribution of the energy carried by EW. For higher order waves the energy flux does not tend to zero at $x_3 \to -\infty$ because of the radiation of energy described above. In general, Figure 3 shows that by choosing appropriate frequencies and loads one can use EW system for detecting surface breaking cracks of different sizes.

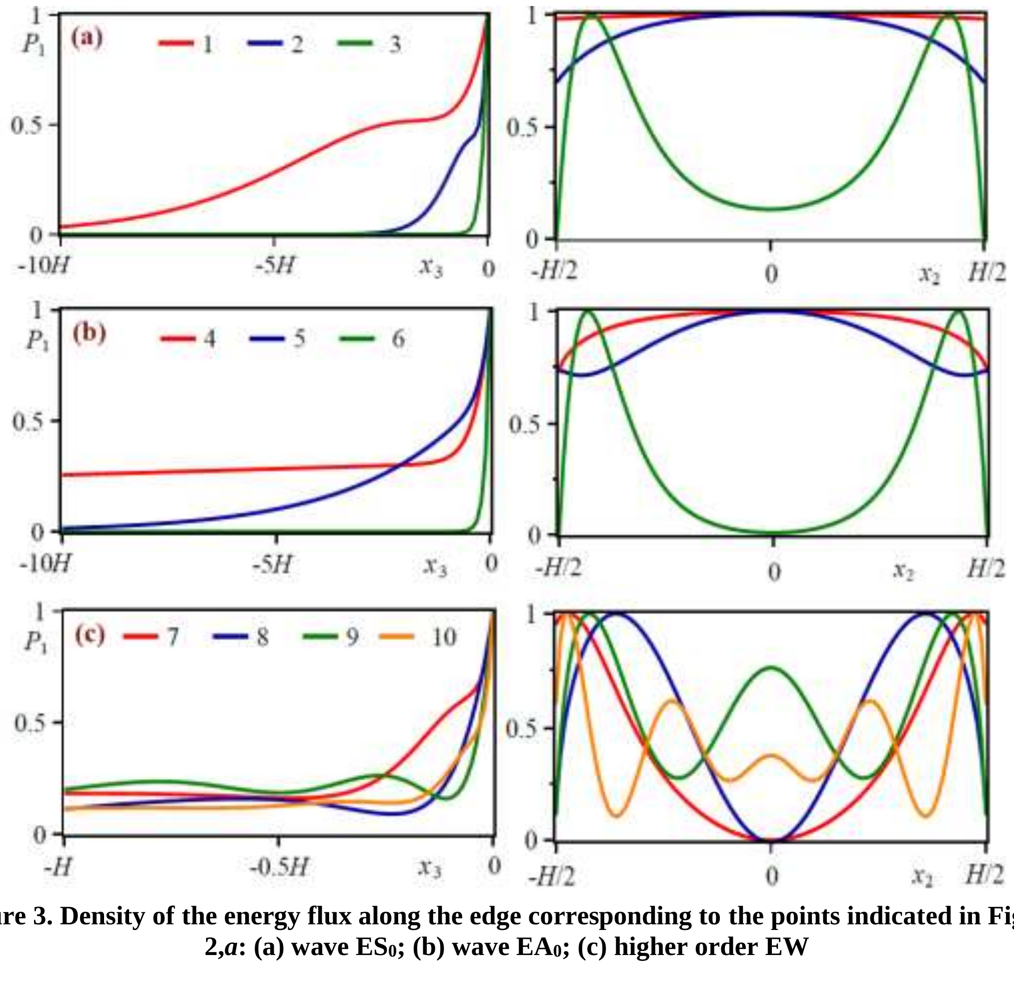

Figure 3. Density of the energy flux along the edge corresponding to the points indicated in Figure 2,*a*: (a) wave ES₀; (b) wave EA₀; (c) higher order EW

All the considerations above are related to BC (4) and (6). A brief summary for the other cases is given below.

- In the case of BC (5) there are no fundamental waves. The higher order waves have the same properties as in the case of free faces but the additional waves $ES_{0.5}$ and $EA_{0.5}$ do not appear. The logarithmical decrements characterizing the attenuation because of radiation to infinity are approximately 10 times less than in the case of free faces.
- The systems of higher order waves exist also in the cases of BC (7) and (8) for both free (BC (4)) and fixed (BC (5)) faces. In general, their properties are similar to these described above for the case of a free edge.
- In the case of BC (4), (7) symmetric fundamental wave exists. The group velocity for this wave is shown in Figure 2,*c* (wave ESm_0). The limit velocity as $\xi \to \infty$ is very close to Rayleigh wave speed but do not coincide with it.

4. Edge waves in shells

Theoretical and numerical investigations of EW in a semi-infinite cylindrical shell of relative thickness $H/R = 0.04$ show that in this shell the system of EW exists, which is analogous to that in a plate in many features. Despite the fact that in some cases the forms of EW have no symmetry in respect to mid-surface the notations ES_k and EA_k with k=0, 0.5, 1, 2,... can be used for the shell as well. In Figure 4,*a* the group velocities for the waves ES_0 and EA_0 in the case of BC (9), (10) are presented. One can see that the behaviour of these curves is similar to that for corresponding waves in plates. The significant difference is that the energy flux of EA_0 in a shell is localized only in the vicinity of the outer wedge $r = R + H/2,\ \ z = 0$, and that of ES_0 in the vicinity of the inner wedge $r = R - H/2,\ \ z = 0$. At low frequencies, the results are in a good agreement with those obtained in [9] in the framework of Kirchhoff–Love theory of shells. Another significant difference is that the wave ES_0 becomes attenuated. In [9] it was explained as an effect of the link between tangential and bending motions of the shell. The behaviour of the LD beyond the framework of Kirchhoff–Love theory is shown in Figure 4,b.

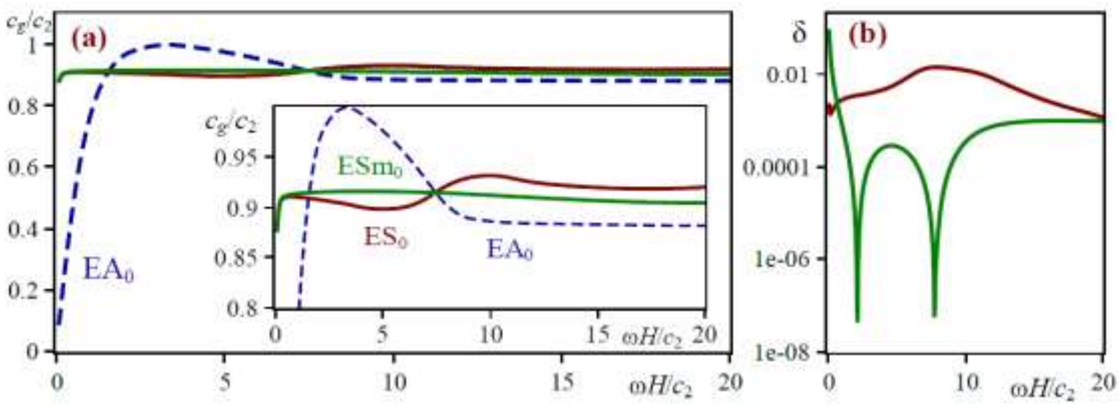

Figure 4. Results for EW in a cylindrical shell: (a) group velocities for the fundamental waves; (b) logarithmical decrement for waves ES_0 and ESm_0

In [9] it was also shown that tangential EW in shells exists if BC on the edge prevent bending vibrations but allow tangential ones. Now it is confirmed on the basis of 3D theory by considering BC (9) and (11). The group velocity and LD for ES_0 in this case can be seen in Figure 4 (wave ESm_0). It is interesting that there are two frequencies at which the attenuation of EW practically vanishes.

The system of higher order EW in a shell of relative thickness under consideration (and below it) is nearly fully analogous to that in a plate even from the quantitative point of view. With some restrictions concerning very high frequencies the dispersion curves can be computed with good accuracy when replacing the shell by a plate with the same metric of the mid-surface. The LD's have the same order as those shown in Figure 2,*b*.

5. Experimental verification

In order to verify theoretical results presented in Section 3 the experiments with the aluminium plate sample with dimensions 600 mm x 400 mm x 4.85 mm were carried out. EW and Lamb waves in the plate were excited by a piezoelectric wafer active sensor (PWAS) (see [7] for more details). The out-of-plane velocities on the surfaces and edge of the plate were measured by means of a Polytec PSV-500 one-dimensional scanning laser Doppler vibrometry. Wavenumber-frequency analysis (WFA) with wavenumber *k* and frequency *f* was performed to investigate dispersion properties of the waves excited by a PWAS adhered to the surface of the specimen [10]. The WFA-images of the signals excited by a broadband pulse loading and measured on the edge are shown in Figure 5. The duration of the pulse was 5μs. Figure 5,*a* presents WFA for the line $x_2 = 0.82$ mm. In Figure 5,*b* WFA of the difference between signals on this line and the line approximately symmetrical with respect to the mid-surface is presented. The theoretical dispersion curves computed with $c_2 = 3175\ \mathrm{m/s}, \nu = 0.33$ are plotted on these graphs for comparison. It can be seen that wave ES_0 is predominant but EA_0 is also present and can be clearly seen in Figure 5,*b*. Moreover, one can see higher order wave $EA_{0.5}$ here.

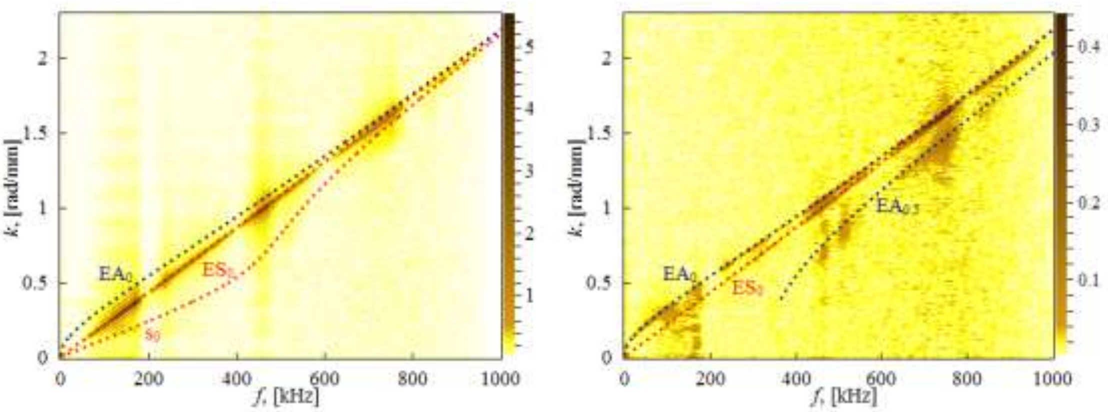

Figure 5. Experimental verification: WFA for pulse loading with theoretical dispersion curves

From the theoretical investigations presented in Section 3 follows that in the plate of 4.85 mm thickness the attenuation of higher order EW is very strong in the frequency range up to 1MHz, so the experimental results presented in Figure 5 agree with the theoretical predictions. It could be expected that these waves will be easier detected in a thicker plate. Also, the experience in numerical searching of them suggests using a load which is distributed non-uniformly along the thickness coordinate.

6. Conclusions

The investigations presented above show that there is a rich system of EW in plates and shells, including waves with different laws of energy flux distribution along thickness coordinate. Thus, it is possible to detect smaller defects than in the case when only first EW described by plate theories are considered. The higher order EW have a strong attenuation that may result in the more challenging implementation of corresponding techniques. However, the fundamental waves can be used effectively in this case. It could be also noticed that detecting of edge defects employing EW can be performed with the equipment developed for using of Lamb waves for damage detection in the interior of the plate.

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